Extended force density method and its expressions

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Abstract

The objective of this work can be divided into two parts. The first one is to propose an extension of the force density method (FDM)[2], a form-finding method for prestressed cable-net structures. The second one is to present a review of various form-finding methods for tension structures, in the relation with the extended FDM.

In the first part, it is pointed out that the original FDM become useless when it is applied to the prestressed structures that consist of combinations of both tension and compression members, while the FDM is usually advantageous in form-finding analysis of cable-nets. To eliminate the limitation, a functional whose stationary problem simply represents the FDM is firstly proposed. Additionally, the existence of a variational principle in the FDM is also indicated. Then, the FDM is extensively redefined by generalizing the formulation of the functional. As the result, the generalized functionals enable us to find the forms of tension structures that consist of combinations of both tension and compression members, such as tensegrities and suspended membranes with compression struts.

In the second part, it is indicated the important role of three expressions used by the description of the extended FDM, such as stationary problems of functionals, the principle of virtual work and stationary conditions using ∇ symbol. They can be commonly found in general problems of statics, whereas the original FDM only provides a particular form of equilibrium equation. Then, to demonstrate the advantage of such expressions, various form-finding methods are reviewed and compared. As the result, the common features and the differences over various form-finding methods are examined. Finally, to give an overview of the reviewed methods, the corresponding expressions are shown in the form of three tables.

Keywords: Form-finding, Tensegrity, Suspended Membrane, Force Density Method, Variational Principle, Principle of Virtual Work

1. Introduction

This is a revised version of [1].

The objective of the first half of this work is to propose an extension of the force density method (FDM)[2], a form-finding method for prestressed cable-net structures. Particularly, for the prestressed tension structures, form-finding is a process to ensure them to have a prestress state, because the existence of a prestress state highly depends on the form of the tension structure.

In section 2, the original FDM is described with its major advantage in form-finding process of cable-net structures. In addition, it is pointed out that the FDM become useless when it is applied to the prestressed structures that consist of combinations of both tension and compression members, e.g. tensegrities. Therefore, the FDM has a scope for extension.

In section 3, a functional whose stationary problem simply represents the original FDM is firstly proposed. Additionally, the existence of a variational principle in the FDM is also indicated, although the formulations provided by the original FDM look different from those related to the variational principle. The clarified functional enables an extension of the FDM.

In section 4, the FDM is extensively redefined by generalizing the formulation of the functional. As the result, the generalized functionals enable us to find the forms of tension structures that consist of combinations of both tension and compression members, such as tensegrities and suspended membranes with compression struts. Moreover, it is pointed out that various functionals can be selected for the purpose of form-finding.

In section 5, some numerical examples of the extended FDM are illustrated to show that the newly introduced functionals enable us to find the forms of tension structures that consist of combinations of both tension and compression members, such as tensegrities and suspended membranes with compression struts.

In section 6, in which the second half of this work is described, it is firstly indicated the important role of three expressions used by the description of the extended FDM, such as stationary problems of functionals, the principle of virtual work and stationary conditions using ∇ symbol. They can be commonly found in general problems of statics, while the original FDM only provides a particular form of equilibrium equation. Then, to demonstrate the advantage of such expressions, various form-finding methods are reviewed and compared. As the result, the common features and the differences over various form-finding methods can be examined. Finally, to give an overview of the reviewed methods, the expressions correspond-

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ing to them are shown in the form of three tables.

2. Force Density Method

2.1. Original Formulation

The FDM is one of the form-finding methods for cable-net structures which was first proposed by *H. J. Schek* and *K. Linkwitz* in 1973. When it is explained, two unique points are usually pointed. The first one is the definition of the force density and the second one is the linear form of the equilibrium equation provided by the FDM.

As the first one, the force density q_j is defined by

$$q_j = n_j/L_j, \tag{2.1}$$

where n_j and L_j denote the tension and length of the j-th member of a structure respectively, as shown in Fig. 2.1(a). In the FDM, each tension member is assigned a positive force density as a prescribed parameter, even though n_j and L_j are unknown. However, In Ref. [2], there is no mention of method to determine them. Then, it is sometimes pointed out that some trials must be carried out to obtain an appropriate set of force densities.

As the second one, although the form-finding problems usually formulated as a non-linear problem, the self-equilibrium equation provided by the FDM is formulated as a set of simultaneous linear equations. In detail, when the force densities and the coordinates of the fixed nodes are prescribed, the self-equilibrium equation of a cable-net structure is expressed as follows:

$$\begin{aligned} \boldsymbol{D} \cdot \boldsymbol{x} &= -\boldsymbol{D}_f \cdot \boldsymbol{x}_f, \\ \boldsymbol{D} \cdot \boldsymbol{y} &= -\boldsymbol{D}_f \cdot \boldsymbol{y}_f, \\ \boldsymbol{D} \cdot \boldsymbol{z} &= -\boldsymbol{D}_f \cdot \boldsymbol{z}_f, \end{aligned} \tag{2.2}$$

where $\boldsymbol{D}$ is the equilibrium matrix and $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are the column vectors containing the coordinates of the nodes. The terms with the subscript f refer to the fixed nodes, whereas those with no subscript are for the free nodes.

Using the inverse matrix of $\boldsymbol{D}$, the nodal coordinates of the free nodes can be simply obtained as follows:

$$\begin{aligned} \boldsymbol{x} &= -\boldsymbol{D}^{-1}\left(\boldsymbol{D}_f \cdot \boldsymbol{x}_f\right), \\ \boldsymbol{y} &= -\boldsymbol{D}^{-1}\left(\boldsymbol{D}_f \cdot \boldsymbol{y}_f\right), \\ \boldsymbol{z} &= -\boldsymbol{D}^{-1}\left(\boldsymbol{D}_f \cdot \boldsymbol{z}_f\right), \end{aligned} \tag{2.3}$$

because, in Eq. (2.2), only $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ contain the unknown variables.

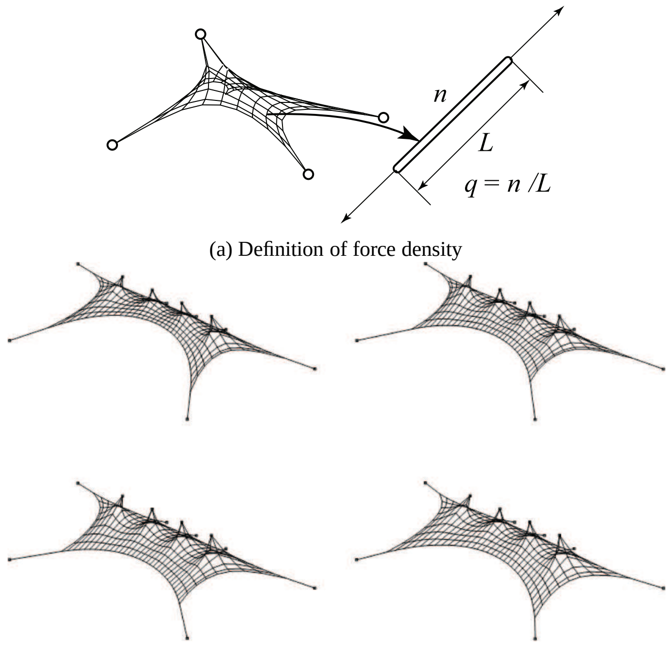
Because Eq. (2.3) simply represents the common procedure to solve a set of simultaneous linear equations, the FDM can be easily implemented by general numerical environments. This can be a major advantage in form-finding analysis of cable-net structures.

Once the nodal coordinates are obtained, the tension in each cable is calculated by using Eq. (2.1). The obtained set of tension represents a self-equilibrium state of the form, i.e.

$$\boldsymbol{n} = \{q_1 L_1, \cdots, q_m L_m\}, \tag{2.4}$$

where m denotes the number of the members. Generally, such a form is called a self-equilibrium form and can be used as a prestressed structure.

Using the FDM, as shown in Fig. 2.1(b), the form of a cable-net can be varied by varying the prescribed coordinates of the fixed nodes and the force densities of the cables.

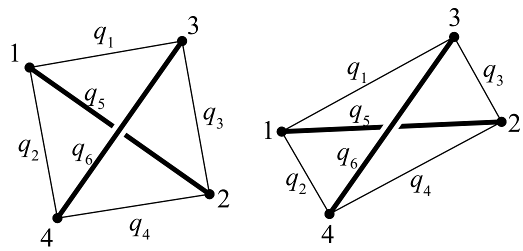


(a) Definition of force density

(b) Form-finding analysis using FDM

Figure 2.1: Force Density Method

2.2. Limitation of FDM



(a) (b)

Figure 2.2: *X-Tensegrities*

In this subsection, the limitation of the FDM is discussed. When it is applied to self-equilibrium systems that consist of a combination of both tension and compression members, e.g. tensegrities, some difficulties arise.

In detail, although it seems possible to assign negative force densities to the compression members and positive force densities to the tension members, the FDM can not keep its conciseness any longer as discussed below.

Let us consider form-finding of a prestressed structure which is called *X-Tensegrity*. Two different forms of *X-Tensegrity* are shown by Fig. 2.2 (a) and (b). An *X-Tensegrity* is a planar prestressed structure that consists of 4 cables (tension) and 2 struts

(compression). As in the case of general tensegrities, the cables connect the struts and the struts do not touch each other.

For such self-equilibrium systems, due to the absence of the fixed nodes, Eq. (2.2) reduces to a simpler form:

$$\boldsymbol{D}\cdot\boldsymbol{x}=\boldsymbol{0},\ \ \boldsymbol{D}\cdot\boldsymbol{y}=\boldsymbol{0},\ \ \boldsymbol{D}\cdot\boldsymbol{z}=\boldsymbol{0}. \tag{2.5}$$

When $\boldsymbol{D}$ is a regular matrix, because it is obvious that $\boldsymbol{D}^{-1}\cdot\boldsymbol{0}=\boldsymbol{0}$, only the trivial solution, i.e.

$$\boldsymbol{x}=\boldsymbol{0},\ \ \boldsymbol{y}=\boldsymbol{0},\ \ \boldsymbol{z}=\boldsymbol{0}, \tag{2.6}$$

is obtained, which implies that every nodes meet at one point, namely $[0,0,0]$.

On the other hand, when $\boldsymbol{D}$ is a singular matrix, i.e. $\det\boldsymbol{D}=0$, Eq. (2.2) generally has complementary solution, which states the possible forms of the structure. Such solutions are obtained by analyzing the null space of $\boldsymbol{D}$. Various methods to analyze the null space of $\boldsymbol{D}$. Various methods have been proposed to analyze such a space (see Ref.[3–6]). However, even if the complementary solutions can be obtained by such methods, the major advantage of the FDM, that the equilibrium equation can be simply solved by inverse matrix, vanishes.

Let us see a simple example, the form-finding analysis of *X-Tensegrity* which is shown by Fig. 2.2. When the FDM is applied to this type of structure, $\boldsymbol{D}$ is calculated by

$$\boldsymbol{D}=\boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}, \tag{2.7}$$

$$\boldsymbol{C}=\begin{bmatrix}1&0&-1&0\\1&0&0&-1\\0&1&-1&0\\0&1&0&-1\\1&-1&0&0\\0&0&1&-1\end{bmatrix}, \tag{2.8}$$

$$\boldsymbol{Q}=\begin{bmatrix}q_1&0&0&0&0&0\\0&q_2&0&0&0&0\\0&0&q_3&0&0&0\\0&0&0&q_4&0&0\\0&0&0&0&q_5&0\\0&0&0&0&0&q_6\end{bmatrix}, \tag{2.9}$$

where $\boldsymbol{C}$ is the branch-node matrix (see Ref.[2] for more detail), $q_1,\cdots,q_4$ are the prescribed force densities of the cables, and q_5,q_6 are of the struts. Then $\boldsymbol{D}$ is represented by:

$$\boldsymbol{D}=\begin{bmatrix}-q_1-q_2-q_5&q_5&q_1&q_2\\q_5&-q_3-q_4-q_5&q_3&q_4\\q_1&q_3&-q_1-q_3-q_6&q_6\\q_2&q_4&q_6&-q_2-q_4-q_6\end{bmatrix}. \tag{2.10}$$

Based on Eq. (2.10), the detail of the form-finding analysis of *X-Tensegrity* is as follows:

- When the assigned force densities, $q_1,\cdots,q_6$, are in the proportion 1:1:1:1:-1:-1, $\boldsymbol{D}$ becomes a singular matrix having 3 dimensional null-space. Then, many solutions are obtained. The components of $\boldsymbol{D}$ and the corresponding complementary solution are as follows:

$$\boldsymbol{D}=\begin{bmatrix}-1&-1&1&1\\-1&-1&1&1\\1&1&-1&-1\\1&1&-1&-1\end{bmatrix}, \tag{2.11}$$

$$(\{q_1,\cdots,q_6\}=\{1,1,1,1,-1,-1\}) \tag{2.12}$$

$$\boldsymbol{x}=a\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\\-1\end{bmatrix},$$

$$\boldsymbol{y}=d\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+e\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+f\begin{bmatrix}0\\0\\1\\-1\end{bmatrix},$$

$$\boldsymbol{z}=g\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+h\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+i\begin{bmatrix}0\\0\\1\\-1\end{bmatrix},$$

  where $a,\cdots,i$ are arbitrary real numbers. This implies, for example, that both Fig. 2.2(a) and (b) satisfy Eq. (2.5). The first terms of the right hand sides denote the position of the center point, namely $[a,d,g]$, and the other terms state some symmetries that all the solutions must have. Note that the particular solution is just $\boldsymbol{x}=\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{0}$.

- When the assigned force densities, $q_1,\cdots,q_6$, are not in the proportion 1:1:1:1:-1:-1, $\boldsymbol{D}$ also becomes a singular matrix but having only 1 dimensional null-space. For example, if the force densities are in the proportion 2:2:2:2:-1:-1, the components of $\boldsymbol{D}$ and the corresponding complementary solution are as follows:

$$\boldsymbol{D}=\begin{bmatrix}-3&-1&2&2\\-1&-3&2&2\\2&2&-3&-1\\2&2&-1&-3\end{bmatrix}, \tag{2.13}$$

$$(\{q_1,\cdots,q_6\}=\{2,2,2,2,-1,-1\}) \tag{2.14}$$

$$\boldsymbol{x}=a\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ \ \boldsymbol{y}=b\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ \ \boldsymbol{z}=c\begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

  where a,b,c are arbitrary. This implies that all the nodes meet at one point, namely$[a,b,c]$.

3. Variational Principle in the FDM

Let us consider a simple functional

$$\Pi(\boldsymbol{x})=\sum_j w_j L_j^2(\boldsymbol{x}), \tag{3.1}$$

where w_j and L_j denote an assigned positive weight coefficient and a function to give the length of the j-th tension member, respectively. The column vector $\boldsymbol{x}$ represents unknown variables,

which are x, y, and z coordinates of the free nodes. It is generalized as an unknown variable container by

$$\boldsymbol{x} = [x_1 \cdots x_n]^T, \tag{3.2}$$

where n denotes the number of the unknown variables. Note that the coordinates related to the fixed nodes are eliminated from $\boldsymbol{x}$ beforehand and directly substituted in L_j.

Actually, the FDM can be simply represented by Eq. (3.1); the reason is as follows.

Let ∇ be the gradient operator by

$$\nabla f = \frac{\partial f}{\partial \boldsymbol{x}} \equiv \left[\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right], \tag{3.3}$$

which points the direction of the greatest rate of increase of f. Let $\delta\boldsymbol{x}$ be an arbitrary column vector by

$$\delta\boldsymbol{x} \equiv \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix}, \tag{3.4}$$

which is called the variation of $\boldsymbol{x}$. Then, the variation of a function $f(\boldsymbol{x})$ is defined by

$$\delta f(\boldsymbol{x}) \equiv \nabla f \cdot \delta\boldsymbol{x}. \tag{3.5}$$

Taking the variation of Eq. (3.1), the stationary condition of the functional is calculated as follows:

$$\delta\Pi = 0 \Leftrightarrow \sum_j 2w_j L_j \delta L_j = 0 \tag{3.6}$$

$$\Leftrightarrow \sum_j \left(2w_j L_j \nabla L_j \cdot \delta\boldsymbol{x}\right) = 0 \tag{3.7}$$

$$\Leftrightarrow \left(\sum_j 2w_j L_j \nabla L_j\right) \cdot \delta\boldsymbol{x} = 0, \tag{3.8}$$

$$\therefore \sum_j 2w_j L_j \nabla L_j = \boldsymbol{0}. \tag{3.9}$$

In particular case that $\{x_1, \cdots, x_n\}$ represents the Cartesian coordinates of the free nodes, each L_j may defined by the following form:

$$\begin{aligned} &L_j\left(p_x,\ p_y,\ p_z,\ q_z,\ q_y,\ q_z\right) \\ &\equiv \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2 + (p_z - q_z)^2}, \end{aligned} \tag{3.10}$$

where p, q denote two ends of j-th member and $p_x, \cdots, q_z$ denote 6 coordinates chosen from $\{x_1, \cdots, x_n\}$. In this case, ∇L_j represents two normalized vectors attached to both ends of j-th member, as shown in Fig. 3.1(a).

On the other hand, suppose the same member resisting two nodal forces applied to both ends, as shown in Fig. 3.1(b). If the magnitude of the tension of the member is denoted by n_j, then the magnitudes of the two nodal forces are also n_j.

By comparing Fig. 3.1(a) and (b), a general form of the self-equilibrium equation for prestressed cable-net structures is obtained as

$$\sum_j n_j \nabla L_j = \boldsymbol{0}. \tag{3.11}$$

To obtain another general form, taking the inner product of Eq. (3.11) with $\delta\boldsymbol{x}$, the **Principle of Virtual Work** for such structures is obtained as

$$\delta w = \sum_j n_j \delta L_j = 0, \tag{3.12}$$

where δL_j is the variation of L_j.

When a set of n_j, i.e.

$$\boldsymbol{n} = \{n_1, \cdots, n_m\}, \tag{3.13}$$

where m denotes the number of the members, satisfies Eq. (3.11), such a set of n_j represents a self-equilibrium state of the structure.

Remembering the definition of the force density, namely Eq.(2.1), Eq. (3.11) can be rewritten as

$$\sum_j q_j L_j \nabla L_j = \boldsymbol{0}, \tag{3.14}$$

which is an alternative form of equilibrium equation provided by the FDM.

Comparing Eq. (3.9) and Eq. (3.14), when Eq. (3.9) is considered as a equilibrium equation, w_j is just a half of q_j. Moreover, when Eq. (3.1) is stationary with a form, it is also the result obtained by the FDM when the prescribed distribution of q_j is as same as w_j.

Therefore, Eq. (3.1), whose stationary condition is Eq. (3.9), is one of the functionals that simply represents the FDM. In addition, it is assumed that the assigned weight coefficients would play the same role in form-finding analysis as the force densities in the FDM.

Because the left hand side of Eq. (3.14) simply represents the gradient of Eq. (3.1), the stationary problem of Eq. (3.1) can be solve by general direct minimization approach, such as the steepest decent method or the dynamic relaxation method [10–13].

Although Eq. (2.2) and Eq. (3.14) look very different, they are accurately identical when each function L_j is defined by Eq. (3.10). Then, let us examine Eq. (3.14) for further comprehension of the linear form of equilibrium equation provided by the FDM. If the non-zero components of ∇L_j is split out as

$$\hat{\nabla} L_j \equiv \left[\ \frac{\partial L_j}{\partial p_x}\ \ \frac{\partial L_j}{\partial p_y}\ \ \frac{\partial L_j}{\partial p_z}\ \ \frac{\partial L_j}{\partial q_x}\ \ \frac{\partial L_j}{\partial q_y}\ \ \frac{\partial L_j}{\partial q_z}\ \right], \tag{3.15}$$

the components of $\hat{\nabla} L_j$ are calculated as

$$\hat{\nabla} L_j = \left[\ \frac{p_x - q_x}{L_j(\boldsymbol{x})}\ \ \frac{p_y - q_y}{L_j(\boldsymbol{x})}\ \ \frac{p_z - q_z}{L_j(\boldsymbol{x})}\ \ \frac{q_x - p_x}{L_j(\boldsymbol{x})}\ \ \frac{q_y - p_y}{L_j(\boldsymbol{x})}\ \ \frac{q_z - p_z}{L_j(\boldsymbol{x})}\ \right]. \tag{3.16}$$

Here, it can be noticed that $L_j(\boldsymbol{x})$ makes $\hat{\nabla} L_j$ non-linear. Then, a linear form can be obtained by multiplying $\hat{\nabla} L_j$ with $L_j(\boldsymbol{x})$,

hence,

$$
\begin{aligned}
&L_j \hat{\nabla} L_j \\
&= \begin{bmatrix} p_x - q_x & p_y - q_y & p_z - q_z & q_x - p_x & q_y - p_y & q_z - p_z \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ q_x \\ q_y \\ q_z \end{bmatrix}, \qquad (3.17)
\end{aligned}
$$

which is the foundation of the linear form of the equilibrium equation provided by the FDM.

Let us consider a case that each variable x_i is also a function of another set of variables $\{y_1, \cdots, y_n\}$, i.e.

$$
\begin{aligned}
x_1 &= x_1(y_1, \cdots, y_n) \\
&\vdots \\
x_n &= x_n(y_1, \cdots, y_n).
\end{aligned} \qquad (3.18)
$$

In this case, the variation of $\boldsymbol{x}$ is given by

$$\delta \boldsymbol{x} = \boldsymbol{D} \cdot \delta \boldsymbol{y}, \qquad (3.19)$$

where

$$\boldsymbol{D} \equiv \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{bmatrix}. \qquad (3.20)$$

On the other hand, the relation between two types of gradients, namely with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$, is given by

$$\begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1} \end{bmatrix} \cdot \boldsymbol{D}. \qquad (3.21)$$

Therefore,

$$
\begin{aligned}
\delta f &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1} \end{bmatrix} \cdot \delta \boldsymbol{x} & (3.22) \\
&= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1} \end{bmatrix} \boldsymbol{D} \delta \boldsymbol{y} & (3.23) \\
&= \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_1} \end{bmatrix} \cdot \delta \boldsymbol{y}, & (3.24)
\end{aligned}
$$

which implies that the expressions such as Eq. (3.9), Eq. (3.11), Eq. (3.12) and Eq. (3.14) remain valid when $\{x_1, \cdots, x_n\}$ represents other coordinate, such as the polar coordinate.

In this fashion, Eq.(3.14) is the general form of the equilibrium equation provided by the FDM. On the other hand, the equilibrium equation provided by the original FDM is one of the particular forms of the general form, which is only valid for the Cartesian coordinate.

Taking the inner product of Eq. (3.14) with $\delta \boldsymbol{x}$, the **Principle of Virtual Work** for the FDM is obtained as

$$\delta w = \sum_j q_j L_j \delta L_j = 0. \qquad (3.25)$$

Similarly, the **Principle of Virtual Work** is also deduced from Eq. (3.9) as:

$$\delta w = \sum_j 2 w_j L_j \delta L_j = 0. \qquad (3.26)$$

As the result, as well as in the general problems of statics, the variational principle for the FDM is simply represented by

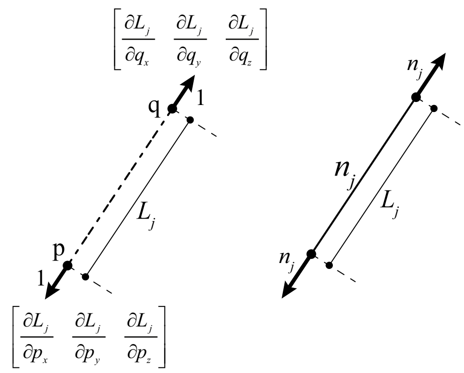
$$\delta \Pi = 0 \qquad (3.27)$$

where $\delta \Pi$ is defined by

$$\delta \Pi \equiv \nabla \Pi \cdot \delta \boldsymbol{x}. \qquad (3.28)$$

To conclude this section, it is important to note that, in the original reference[2], Eq. (3.1) have been mentioned by the following theorem:

> *"THEOREM 1. Each equilibrium state of an unloaded network structure with force densities q_j is identical with the net, whose sum of squared way lengths weighted by q_j is minimal. "*

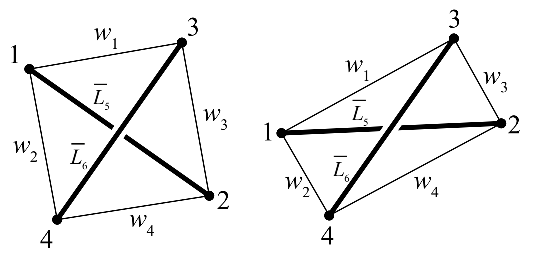


(a) ∇L_j (b) Equilibrium state

Figure 3.1: Linear Member

4. Extended Force Density Method

4.1. Generalized Formulation of Functional



(a) (b)

Figure 4.1: *X-Tensegrities and Prescribed Parameters*

In this subsection, the FDM is extended for form-finding of structures that consist of combinations of both tension and compression members, e.g. tensegrities.

Let us reconsider the form-finding of *X-Tensegrity* again. Although it seems possible to assign negative weight coefficients to the compression members and positive weight coefficients to the tension members, the same difficulties which is

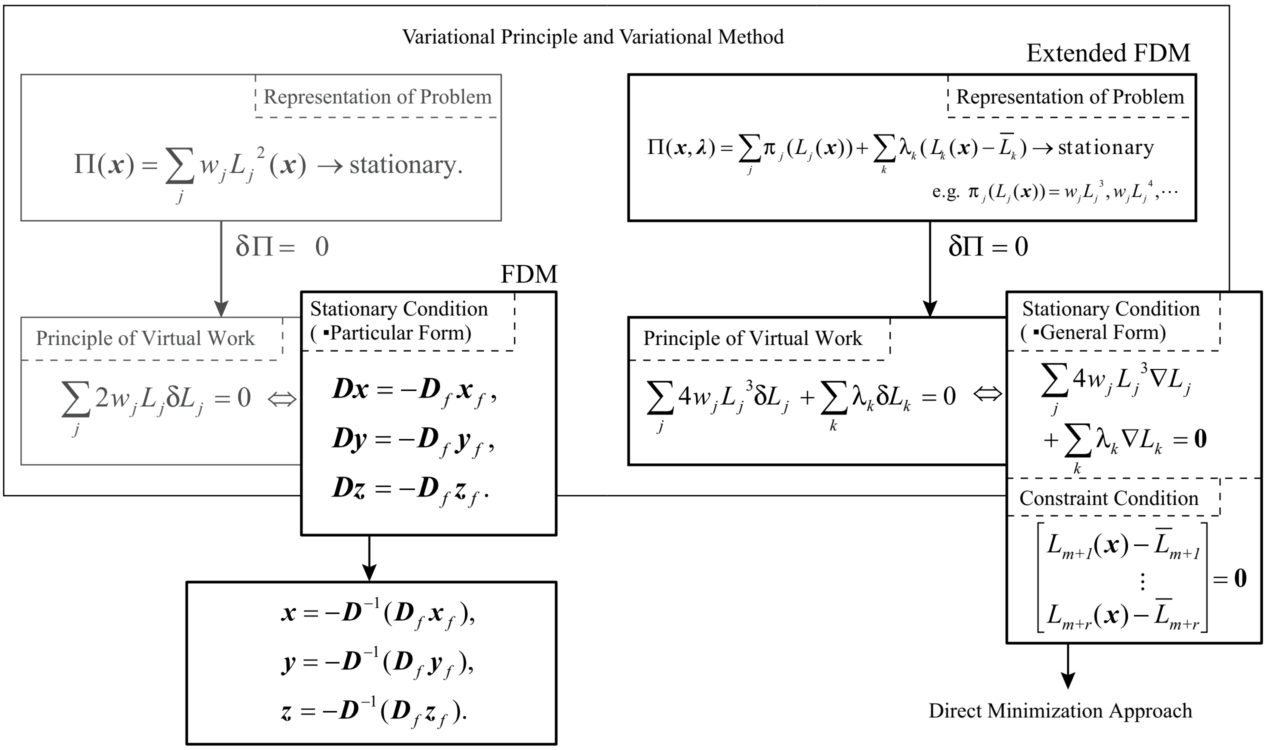


Figure 4.2: Relation between Original FDM and Extended FDM

pointed out in subsection 2.2 also arise from the stationary problem of $\Pi(\boldsymbol{x}) = \sum w_j L_j^2(\boldsymbol{x})$. In detail, when the assigned weight coefficients w_j are in the proportion 1:1:1:1:-1:-1, for the 4 tension members and the 2 compression members respectively, the stationary points forms a space. On the other hand, when w_j are not int the proportion 1:1:1:1:-1:-1, the stationary points vanish.

First, it is obvious that, without no constraint conditions, every lengths of the members become simultaneously 0 or infinite. This is due to the absence of information about the scale of the structure. Remembering that, in the original FDM, such information is given by the prescribed coordinates of the fixed nodes, let the lengths of the compression members be prescribed. Then, using the *LagrangeÅfs multiplier method*, a modified functional is obtained as

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_j w_j L_j^2(\boldsymbol{x}) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right), \tag{4.1}$$

where the first sum is taken for all the tension members and the second is for all the compression members. In addition, λ_k and $\bar{L}_k$ denote the *LagrangeÅfs multiplier* and the prescribed length of the k-th compression member, respectively. Note that the positive weight coefficients w_j are assigned to only the tension members and the prescribed lengths $\bar{L}_k$ are assigned to only the compression members as shown in Fig.4.1.

However, Eq. (4.1) does not completely eliminate the above mentioned difficulties. For example, if the assigned weight coefficients of the tension members $w_1, \cdots, w_4$ are in the proportion 1:1:1:1, and the prescribed lengths of the compression members $\bar{L}_5, \bar{L}_6$ are in the proportion 1:1, both Fig. 4.1(a) and (b) satisfy the stationary condition of Eq. (4.1). By using the Pythagorean theorem, i.e. $c^2 = a^2 + b^2$, it can be easily verified that the sum of squared lengths of the tension members takes the same value for both Fig. 4.1(a) and (b). Then, it is assumed that such difficulties depend on the power of L_j, i.e. 2.

Thus, other functionals, such as

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_j w_j L_j^4(\boldsymbol{x}) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right), \tag{4.2}$$

are introduced, because it is possible to use other powers of L_j instead of 2.

Solving the stationary problem of Eq. (4.2), Fig. 4.1 (a) becomes the unique solution when the weight coefficients of the tension members $w_1, \cdots, w_4$ and the prescribed lengths of the compression members $\bar{L}_5, \bar{L}_6$ are in the proportion 1:1:1:1 and 1:1 respectively. On the other hand, when they are 1:8:8:1 and 1:1, Fig. 4.1 (b) becomes the unique solution. Actually, Fig. 4.1(a) and (b) are the real numerical results obtained by solving such problems.

By the way, let us discuss the following general formulation of functional:

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_j \pi_j \left(L_j(\boldsymbol{x})\right) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right). \tag{4.3}$$

The stationary condition of Eq. (4.3) with respect to $\boldsymbol{x}$ is as follows:

$$\frac{\partial \Pi}{\partial \boldsymbol{x}} = \sum_j \frac{\partial \pi_j \left(L_j(\boldsymbol{x})\right)}{\partial L_j} \nabla L_j + \sum_k \lambda_k \nabla L_k = \boldsymbol{0}. \tag{4.4}$$

Because Eq. (4.4) has the same form of Eq. (3.11), it can be considered as a equilibrium equation. Then, when Eq. 4.3 is stationary, the following non-trivial set of axial forces must satisfy the general form of equilibrium equation:

$$\{n_1, \cdots, n_{m+r}\} = \left\{ \frac{\partial \pi_1}{\partial L_1}, \quad \cdots \quad , \frac{\partial \pi_m}{\partial L_m}, \quad \lambda_{m+1}, \quad \cdots \quad , \lambda_{m+r} \right\}, \tag{4.5}$$

which represents a self-equilibrium state of structure, where m and r denote the numbers of the tenion and the compression members respectively.

On the other hand, the stationary condition of Eq. (4.3) with respect to $\boldsymbol{\lambda}$ is given by

$$\frac{\partial \Pi}{\partial \boldsymbol{\lambda}} = \begin{bmatrix} \frac{\partial \Pi}{\partial \lambda_{m+1}} \\ \vdots \\ \frac{\partial \Pi}{\partial \lambda_{m+r}} \end{bmatrix} = \begin{bmatrix} L_{m+1}(\boldsymbol{x}) - \bar{L}_{m+1} \\ \vdots \\ L_{m+r}(\boldsymbol{x}) - \bar{L}_{m+r} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.6}$$

Therefore, any functional that compatible to Eq. (4.3) has a possibility to be used for such form-finding problems. From now on, let us call π_j the element functional. Then the following policy is proposed:

- Perform form-finding analysis by solving a stationary problem that is formulated by freely selected element functionals.

Taking the inner product of Eq. (4.4) with $\delta \boldsymbol{x}$, the **Principle of Virtual Work** is obtained as:

$$\delta w = \sum_j \frac{\partial \pi_j \left(L_j(\boldsymbol{x}) \right)}{\partial L_j} \delta L_j + \sum_k \lambda_k \delta L_k = 0. \tag{4.7}$$

Additionally, replacing the partial derivatives in Eq. 4.7 by n_j, the following form can be also used as the **Principle of Virtual Work** for general prestressed structures that consist of combinations of both tension and compression members:

$$\delta w = \sum_j n_j(L_j) \delta L_j + \sum_k \lambda_k \delta L_k = 0. \tag{4.8}$$

Comparing Eq. (4.7) and Eq. (4.8), if $w_j L_j^2$ is selected as the element functional, the following relations are derived:

$$n_j = \frac{\partial w_j L_j^2}{\partial L_j} = 2 w_j L_j, \therefore w_j = n_j / 2L_j. \tag{4.9}$$

Hence, w_j can be considered as a half of the force density of the j-th member.

On the other hand, if $w_j L_j^4$ is selected, then,

$$n_j = \frac{\partial w_j L_j^4}{\partial L_j} = 4 w_j L_j^3, \therefore w_j = n_j / 4L_j^3. \tag{4.10}$$

Thus, in this fashion, various quantities that are similar to the force density can be defined. Then, let us call the new quantities, such as $w_j = n_j / 4L_j^3$, the extended force density.

Apartting from the linear form of the equilibrium equation, now, the main characteristics of the original FDM are reconsidered as follows:

- The coordinates of the fixed nodes are prescribed as constraint conditions.
- The force densities $q_j = n_j / L_j$ are assigned to each tension member as known parameters.

On the other hand, for example, when $w_j L_j^4$ is selected as the element functional, the main characteristics of the extended FDM are as follows:

- The coordinates of the fixed nodes and the lengths of the compression members are prescribed as constraint conditions.
- The extended force densities, e.g. $w_j = n_j / 4L_j^3$, are assigned to each tension member as known parameters.

Therefore, the extended FDM can be considered as similar method to the original FDM.

Considering both approach as solving the stationary problems, their main difference is related to the form of the stationary conditions and the selection of the computational methods. In the original FDM, they are as follows:

- The stationary condition of functional is represented by a particular form.
- The stationary condition is simply solved by using an inverse matrix $\boldsymbol{D}^{-1}$.

On the other hand, in the extended FDM, they are as follows:

- The stationary condition of functional is represented by a general form.
- The stationary condition is solved by general direct minimization approaches.

As an overview of the relation between the original and the extended FDM, Fig. 4.2 shows a diagram of both procedures.

4.2. Additional Analyses

In this subsection, some additional numerical analyses are reported to supplement the concept of the extended FDM.

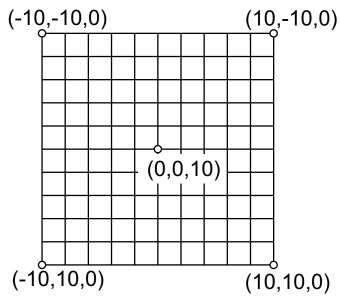
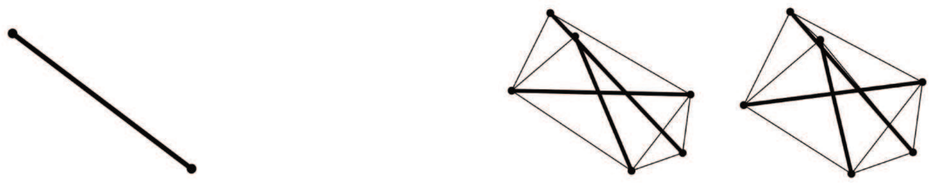


Figure 4.3: Analytical Model

Let us consider a net that consists of 220 cables (tension members) connecting one another and having 5 fixed nodes as shown in Fig. 4.3. The prescribed coordinates of the fixed nodes are also shown in the figure.

(i) $\sum L_j \to \min$ (ii) $\sum L_j^2 \to \min$ (iii) $\sum L_j^3 \to \min$ (iv) $\sum L_j^4 \to \min$

Figure 4.4: Optimization Results of Cable-nets



(i) $\sum L_j \to \min$ (ii) $\sum L_j^2 \to \min$ (iii) $\sum L_j^3 \to \min$ (iv) $\sum L_j^4 \to \min$

Figure 4.5: Optimization Results of *Simplex Tensegrities*

Next, let us find the forms taking minimum numbers of $\sum_j L_j, \cdots, \sum_j L_j^4$, namely

$$\sum_j L_j^p \to \min, (p \in \{1, 2, 3, 4\}), \tag{4.11}$$

where L_j denotes the length of the j-th cable. The results of minimization processes are shown in Fig. 4.4.

On the other hand, Fig. 4.5 shows the other results of the same series of minimization processes performed on another model, which is based on *Simplex Tensegrity*. A *Simplex Tensegrity* is a prestressed structure that consists of 9 cables(tension) and 3 struts(compression). In addition, the minimization processes were only performed on the cables, whereas, the lengths of the struts were kept constant at prescribed length, 10.0, during the processes.

Comparing particularly Fig. 4.4(ii) and Fig. 4.5(ii), $w_j L_j^2$ seems not good for form-finding of tensegrities.

For more detail, when $L_j = 0$, ∇L_j can no be defined because ∇L becomes division by zero (see Eq. (3.16)). Therefore, three of the results, i.e. Fig. 4.4(i), Fig. 4.5(i) and Fig. 4.5(ii), are only the solutions of minimization problems, whereas the others are also the solutions of stationary problems.

5. Numerical Examples

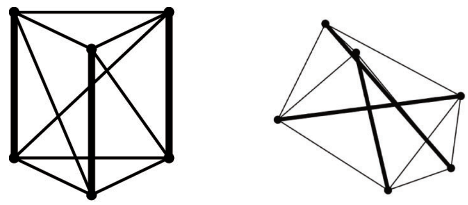
In this section, numerical examples of the extended FDM are presented.

In the examples, the stationary problems are represented in the following form:

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \Pi_w(\boldsymbol{x}) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right). \tag{5.1}$$

Then, for simplicity, the problems were solved by general direct minimization approaches, in which just $\Pi_w(\boldsymbol{x})$ were minimized as objective functions and the lengths of the struts were kept constant at the prescribed lengths $\bar{L}_k$ during each minimization process. Hence, only $\boldsymbol{x}$, or the form, was obtained in each problem.

5.1. Structures Consisting of Cables and Struts



(a) Connection (b) Result

Figure 5.1: Simplex Tensegrity

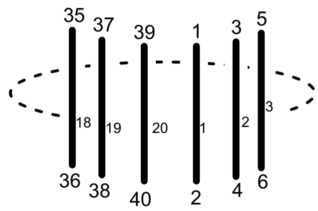


Figure 5.2: Sequential Numbers on Struts

As mentioned in section 4.2, a form of the *Simplex Tensegrity* that consists of 9 cables(tension) and 3 struts(compression) can be obtained by solving the following problem:

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_j L_j^4(\boldsymbol{x}) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right) \to \text{stationary}. \tag{5.2}$$

Here, in the relation with Eq. (5.1), the objective function Π_w is $\sum_j L_j^4(\boldsymbol{x})$.

The **Principle of Virtual Work** corresponding to Eq. (5.2) is as follows:

$$\delta w = \sum_j 4 L_j^3 \delta L_j + \sum_k \lambda_k \delta L_k = 0. \tag{5.3}$$

Cable# (w_1)	Node#	Cable# (w_2)	Node#
1	1-3	41	1-4
2	2-4	42	2-5
⋮	⋮	⋮	⋮
39	39-1	79	39-2
40	40-2	80	40-3

(C_1)

Cable# (w_1)	Node#	Cable# (w_2)	Node#
1	1-5	41	1-6
2	2-6	42	2-7
⋮	⋮	⋮	⋮
39	39-3	79	39-4
40	40-4	80	40-5

(C_2)

⋮

Cable# (w_1)	Node#	Cable# (w_2)	Node#
1	1-19	41	1-20
2	2-20	42	2-21
⋮	⋮	⋮	⋮
39	39-17	79	39-18
40	40-18	80	40-19

(C_9)

Table 1: Connections by Cables

In the analysis, every prescribed lengths of the struts, $\bar{L}_k$, were set to 10.0. The connection between the struts and the cables in a *Simplex Tensegrity* is as shown in Fig. 5.1 (a). The obtained result is shown by Fig 5.1 (b).

Generally, in the direct minimization approaches (see Ref. [10–13]), different initial configurations of $\boldsymbol{x}$ may give different results, because the functionals are basically multimodal.

Then, diffrent random numbers from -2.5 to 2.5 were roughly set to the initial configuration of $\boldsymbol{x}$ in each analysis in order to obtain local minimums as many as possible, because it is not only the global minimum but any local minimum has an ability to be used as a tension structure.

In this example, particularly, only Fig. 5.1 (b) were constantly obtained. However, the same strategy was used in the following examples and in some of them, many local minimums were obtained.

Let us consider more complex tensegrities such as a system that consists of 80 cables (tension) and 20 struts (compression). Let us assign sequential node numbers to all the ends of the struts, as shown in Fig. 5.2.

Even there are a variety of connections between the struts by the cables, 9 of connections were tested. For each connection, the node numbers that each cable connects are as shown in Tab. 1.

In this example, the following stationary problem was formulated and a series of form-finding analyses were carried out:

$$\Pi(\boldsymbol{x},\lambda) = \sum_{j=1}^{40} w_1 L_j^4(\boldsymbol{x}) + \sum_{j=41}^{80} w_2 L_j^4(\boldsymbol{x}) + \sum_k \lambda_k \left(L_k(\boldsymbol{x}) - \bar{L}_k\right) \rightarrow \text{stationary}, \tag{5.4}$$

in which the cables were divided into two groups and w_1 denotes the common weight coefficients for the first group, whereas w_2 is for the second group. In addition, every prescribed length of the struts, $\bar{L}_k$, were constantly set to 10.0.

The **Principle of Virtual Work** corresponding to Eq. (5.4) is as follows:

$$\delta w = \sum_{j=1}^{40} 4 w_1 L_j^3 \delta L_j + \sum_{j=41}^{80} 4 w_2 L_j^3 \delta L_j + \sum_k \lambda_k \delta L_k = 0. \tag{5.5}$$

When $w_1 : w_2 = 1 : 2$, Fig. 5.3 shows the most frequently obtained results for each connection. Fig. 5.4 (j) to (l) shows how the form varied when the proportion between w_1 and w_2 was varied. Interestingly, between Fig. 5.4 (k) and (l), a transition of the form was observed.

It must be noted that the results shown by Fig. 5.3 are just a fraction of various obtained results and a lot of local minimums were obtained for each connection, which implies that the functionals are multimodal. An example of such local minimums are given by Fig. 5.4 (f) and (m). Although both results have exactly the same connection and the prescribed parameters, except the initial configuration of $\boldsymbol{x}$, their forms look completely different. This is due to the random numbers which were set to $\boldsymbol{x}$ in each initial step.

5.2. Structures Consisting of Cables, Membranes and Struts

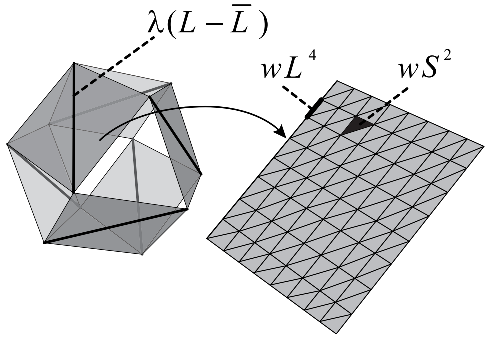


Figure 5.5: Analytical Model

For form-finding of structures that consist of combinations of cables (tension), membranes (tension), and struts (compression), if the cables are represented by a set of linear elements and the membranes, by a set of triangular elements, Eq. (4.3) can be extended as follows:

$$\Pi(\boldsymbol{x},\lambda) = \sum_j \pi_j\left(L_j(\boldsymbol{x})\right) + \sum_k \pi_k\left(S_k(\boldsymbol{x})\right) + \sum_l \lambda_l\left(L_l(\boldsymbol{x}) - \bar{L}_l\right), \tag{5.6}$$

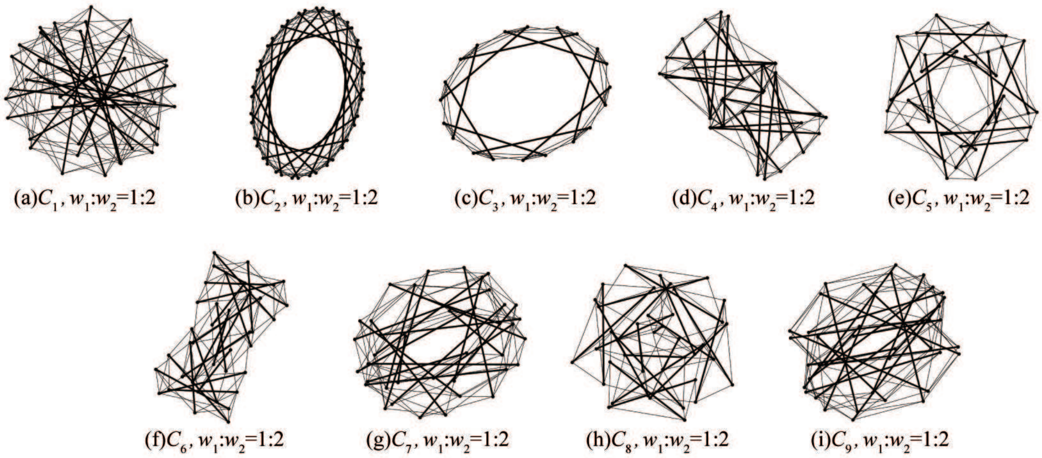


Figure 5.3: Discovered Tensegrities

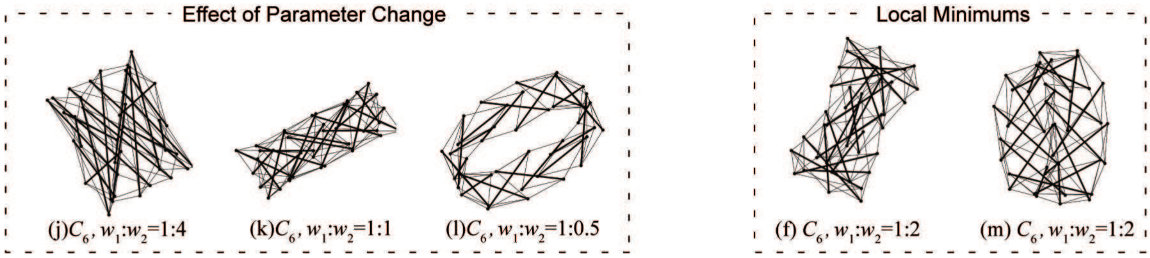


Figure 5.4: Variety of Forms (C_6)

where the first sum is taken for all the linear elements, the second is for all the triangular elements and the third is for all the struts. In addition, L_j and S_k are defined as the functions to give the length of the j-th linear element and the area of the k-th triangular element respectively.

The stationary condition of Eq. (5.6) with respect to $\boldsymbol{x}$ is as follows:

$$\frac{\partial \Pi}{\partial \boldsymbol{x}} = \nabla \Pi = \sum_j \frac{\partial \pi_j \left(L_j(\boldsymbol{x})\right)}{\partial L_j} \nabla L_j + \sum_k \frac{\partial \pi_k \left(S_k(\boldsymbol{x})\right)}{\partial S_k} \nabla S_k + \sum_l \lambda_l \nabla L_l = \boldsymbol{0}.$$

Replacing the partial differential factors by

$$n_j = \frac{\partial \pi_j \left(L_j(\boldsymbol{x})\right)}{\partial L_j}, \ \sigma_k = \frac{\partial \pi_k \left(S_k(\boldsymbol{x})\right)}{\partial S_k}, \tag{5.7}$$

a general form that can be considered as a self-equilibrium equation for such systems is obtained as:

$$\sum_j n_j \nabla L_j + \sum_k \sigma_k \nabla S_k + \sum_l \lambda_l \nabla L_l = \boldsymbol{0}. \tag{5.8}$$

Taking the inner product of Eq. (5.8) with $\delta \boldsymbol{x}$, the **Principle of Virtual Work** corresponding to Eq. (5.8) is obtained as follows:

$$\delta w = \sum_j n_j \delta L_j + \sum_k \sigma_k \delta S_k + \sum_l \lambda_l \delta L_l = 0. \tag{5.9}$$

In order to alter the cables in the tensegrities by tension membranes, a form-finding analysis based on the above formulations was carried out with an analytical model shown by Fig. 5.5. The model is based on the cuboctahedron and consists of 24 cables, 6 membranes, and 6 struts. In detail, every members were translated to purely geometric components such as curves, surfaces and lines, then, each curve were discretized by 8 linear elements and each surface was discretized by 128 triangular elements.

In the analysis, the following stationary problem was formulated and solved:

$$\Pi(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_j w_j L_j^4(\boldsymbol{x}) + \sum_k w_k S_k^2(\boldsymbol{x}) + \sum_l \lambda_l \left(L_l(\boldsymbol{x}) - \overline{L}_l\right) \to \text{stationary}. \tag{5.10}$$

The **Principle of Virtual Work** corresponding to Eq. (5.10) is as follows:

$$\delta w = \sum_j 4 w_j L_j^3 \delta L_j + \sum_k 2 w_k S_k \delta S_k + \sum_l \lambda_l \delta L_l = 0. \tag{5.11}$$

At first, all of the weight coefficients of the linear elements were set to 2.0, those of the triangular elements, 1.0, and the prescribed lengths of the struts, 10.0. Then the initial result shown by Fig. 5.6 (n) was obtained. By varying w_j, w_k and $\overline{L}_l$, the form was able to be varied as shown in Fig. 5.6 (o) to (q).

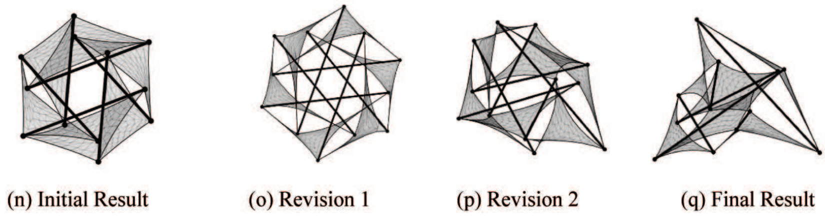


Figure 5.6: Result of Complex Tensegrity

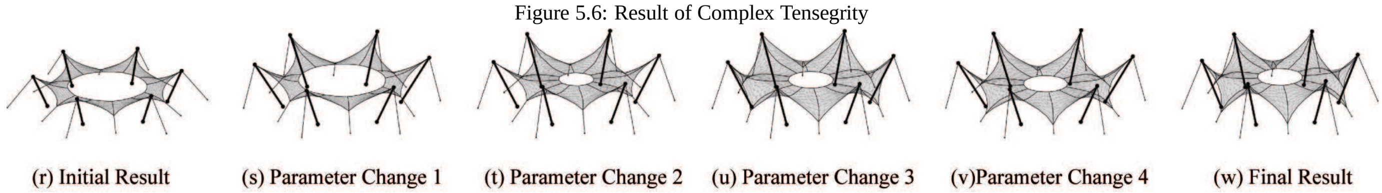


Figure 5.7: Form-finding of Suspended Membrane

5.3. *Structures Consisting of Cables, Membranes, Struts and Fixed Nodes*

A form-finding analysis of a suspended membrane structure based on the famous Tanzbrunnen was carried out. It is located in Cologne (Köln), Germany, and was designed by *F. Otto* (1957).

In the analysis, the following problem was formulated and solved:

$$\Pi(\boldsymbol{x}, \lambda) = \sum_j w_j L_j^4(\boldsymbol{x}) + \sum_k w_k S_k^2(\boldsymbol{x}) + \sum_l \lambda_l \left(L_l(\boldsymbol{x}) - \bar{L}_l \right) \rightarrow \text{stationary}, \tag{5.12}$$

where, as well as in the previous example, the first sum is taken for all the linear elements, the second is for all the triangular elements, and the third is for all the struts. As well as in section 3, the prescribed coordinates of the fixed nodes are eliminated from $\boldsymbol{x}$ beforehand and directly substituted in L_j and S_k.

By varying w_j, w_k and $\bar{L}_l$, as shown in Fig. 5.7, the form was able to be varied. Note that Fig. 5.7(w) looks having a close form to the real one.

6. Review of Various Form-Finding Methods

In the description of the extended FDM, which is just introduced in the previous sections, three diffrent types of expressions are mainly used, they are, stationary problems of functionals, the principle of virtual work, and stationary conditions using ∇ symbol. Such expressions can be commonly found in general problems of statics.

In this section, by using such expressions, various form-finding methods are reviewed and compared, in the relation with the extended FDM. The methods to be reviewed are, the original FDM, the surface stress density method (SSDM) [7], and the methods to solve the minimal surface problem, a variational method for tensegrities [8].

First, let us review the SSDM, which is also an extension of the FDM and a form-finding method for membrane structures. It was proposed by B. Maurin et al., in 1998.

In the SSDM, the membranes are discretized by many triangular membrane elements and in each elements, the Cauchy stress tensor $\sigma^{\alpha}_{\cdot\beta}$ is assumed as uniform and isotropic, i.e. $\sigma^{\alpha}_{\cdot\beta} = \hat{\sigma}\delta^{\alpha}_{\cdot\beta}$, in order to obtain uniform stress surfaces. As an analogy of the definition of the force density, the surface stress density Q_j in each element j is defined by

$$Q_j = \sigma_j / S_j, \tag{6.1}$$

where σ_j is just the scalar multiple of $\hat{\sigma}_j$ with the element thickness t_j and S_j denotes each element area. Then, an equilibrium equation is formulated by considering the equilibrium of all nodes of the triangular elements.

Let us rewrite the equilibrium equation provided by the SSDM by using ∇ symbol, which is the same fashion that applied to the original FDM (see section 3). First, let $S(\boldsymbol{x})$ be a function to give the area of a triangle determined by three nodes whose 9 coordinates are included in $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$. When ∇S is defined by

$$\nabla S \equiv \left[\frac{\partial S}{\partial x_1}, \cdots, \frac{\partial S}{\partial x_n} \right], \tag{6.2}$$

it represents three vectors attached to each node, as shown in Fig. 6.1(a).

By the way, let a triangular membrane element, of which the thickness is assumed as uniform and denoted by t, be resisting three nodal forces applied to each node. For the Cauchy stress filed in each element, in the same fashion of the SSDM, let $\sigma^{\alpha}_{\cdot\beta} = \hat{\sigma}\delta^{\alpha}_{\cdot\beta}$ and $\sigma = \hat{\sigma}t$. When such an element is in equilibrium with the three nodal forces, the nodal forces can be calculated uniquely, and it is as shown in Fig. 6.1 (b).

Comparing Fig. 6.1 (a) and (b), a general form of self-equilibrium equation for general systems that consist of such elements is obtained as

$$\sum_j \sigma_j \nabla S_j = \boldsymbol{0}, \tag{6.3}$$

taking the inner product of Eq. (6.3) with $\delta\boldsymbol{x}$, the **Principle of**

Virtual Work for such a system is obtained as:

$$\delta w = \sum_j \sigma_j \delta S_j = 0. \qquad (6.4)$$

By the way, in the SSDM, the surface stress density Q_j is defined by

$$Q_j = \sigma_j / S_j, \qquad (6.5)$$

then substituting Eq. (6.5) to Eq. (6.3), a general form for the self equilibrium equation of the SSDM is obtained as:

$$\sum_j Q_j S_j \nabla S_j = \mathbf{0}. \qquad (6.6)$$

Then, one of the functionals that simply represents the SSDM is as follows:

$$\Pi(\boldsymbol{x}) = \sum_j w_j S_j^2(\boldsymbol{x}), \qquad (6.7)$$

because the stationary condition of Eq. (6.7) is given by

$$\frac{\partial \Pi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \nabla \Pi = \sum_j 2 w_j S_j \nabla S_j = \mathbf{0}, \qquad (6.8)$$

and when Eq. (6.8) is considered as one of the equilibrium equations given by Eq. (6.3), w_j can be considered as just a half of Q_j. In addition, each w_j also represents an extended force density such as $w_j = \sigma_j / 2S_j$.

Based on the proposed functionals for the original FDM and the SSDM, namely $\sum_j w_j L_j^2$ and $\sum_j w_j S_j^2$, the SSDM looks truly an extension of the original FDM.

Moreover, based on the corresponding **Principle of Virtual Works**, i.e.

$$\delta w = \sum_j 2 w_j L_j \delta L_j = 0, \qquad (6.9)$$

$$\delta w = \sum_j 2 w_j S_j \delta S_j = 0, \qquad (6.10)$$

$2w_j L_j$ and $2w_j S_j$ can be considered as general forces which act within the members or the elements and tend to produce small change of L_j and S_j, respectively.

In addition, if the **Principle of Virtual Works** are written in the following forms:

$$\delta w = \sum_j w_j \delta \left(L_j^2\right) = 0, \qquad (6.11)$$

$$\delta w = \sum_j w_j \delta \left(S_j^2\right) = 0, \qquad (6.12)$$

then, the extended force densities, $w_j = n_j / 2L_j$ and $w_j = \sigma_j / 2S_j$, can be considered as general forces which act within the members or the elements and tend to produce small change of L_j^2 and S_j^2.

Next, let us compare the following two problems:

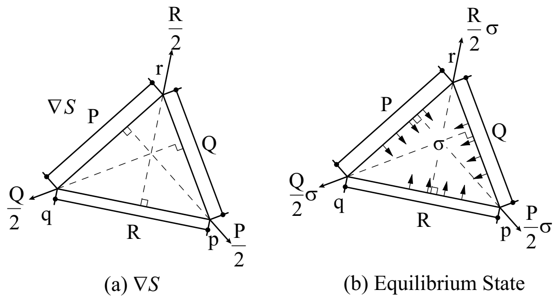


Figure 6.1: Triangular Element

$$\Pi(\boldsymbol{x}) = \sum_j S_j(\boldsymbol{x}) \rightarrow \text{stationary}, \qquad (6.13)$$

$$\Pi(\boldsymbol{x}) = \sum_j S_j^2(\boldsymbol{x}) \rightarrow \text{stationary}, \qquad (6.14)$$

because, for the minimal surface problem, $\sum_j S_j$ is often used, whereas, $\sum_j S_j^2$ simply represents the SSDM when the distribution of the surface stress densities is given as uniform.

By applying both problems to the same numerical model shown in Fig. 6.2, 2 pairs of results were obtained as shown in Fig. 6.3. In addition, such forms are easily observed by a soap-film experiment.

First of all, due to the fact that they are different functionals, it is not obvious that the stationary points given by Eq. (6.14) are minimal surfaces. However, the forms of (a-1) and (b-1) look identical with (a-2) and (b-2). On the other hand, their mesh distributions look dissimilar, i.e. the results given by $\sum_j S_j^2$ seem to have better mesh distributions in comparison with those by $\sum_j S_j$.

Then, let us see the **Principle of Virtual Works**, i.e.

$$\delta w = \sum_j \delta S_j = 0, \qquad (6.15)$$

$$\delta w = \sum_j 2 S_j \delta S_j = 0. \qquad (6.16)$$

Then, it can be noticed that, in Eq. (6.16), the general forces which tend to produce small change of S_j are proportional to S_j, which implies that each element is hard to have bigger or smaller area compared to the surrounding elements (see Fig. 6.4). On the other hand, in Eq. (6.15), whatever element area that each element has, the coefficients of δS_j remain always 1. Therefore, as long as the total element area is minimum, each element is able to have bigger or smaller area compared to the surrounding elements. Thus, the difference appeared in Fig. 6.3 can be well explained by the principle of virtual works.

The SSDM has been proposed for structures that consist of combinations of membranes and cables. When the SSDM is applied to such structures, as same as in section 5.2, the cables are represented by linear elements and the membranes are represented by triangular elements. Then, the force densities are assigned to the linear elements and the surface stress densities are assigned to the triangular elements. In such cases, the SSDM can be simply represented by

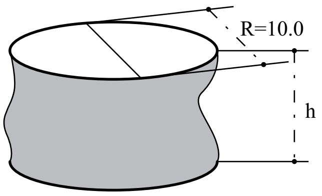
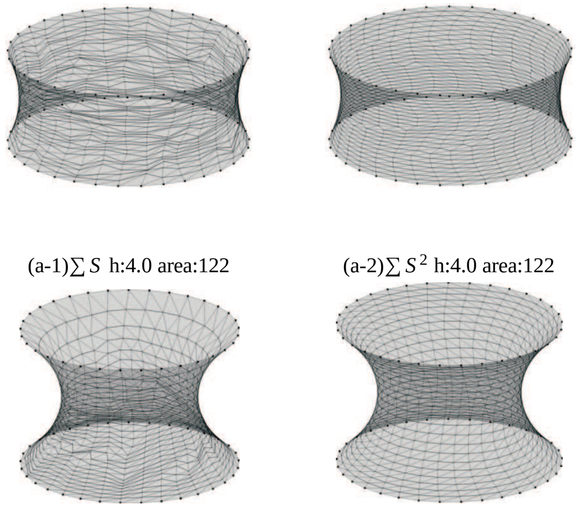


Figure 6.2: Form-finding Problem of Simple Membrane



(a-1)$\sum S$ h:4.0 area:122 (a-2)$\sum S^2$ h:4.0 area:122

(b-1)$\sum S$ h:6.5 area:186 (b-2)$\sum S^2$ h:6.5 area:186

Figure 6.3: Comparison of $\sum S$ and $\sum S^2$

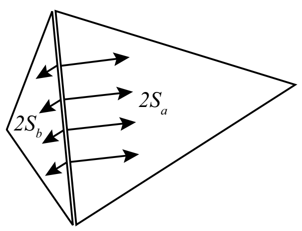


Figure 6.4: Stress State in SSDM

$$\Pi(\boldsymbol{x}) = \sum_j w_j L_j^2(\boldsymbol{x}) + \sum_k w_k S_k^2(\boldsymbol{x}) \rightarrow \text{stationary}, \qquad (6.17)$$

where the first sum is taken for all the linear elements and the second is for all the triangular elements. Fig. 6.5 shows one of the results given by solving Eq. (6.17). The corresponding **Principle of Virtual Work** is as follows:

$$\delta w = \sum_j 2w_j L_j \delta L_j + \sum_k 2w_k S_k \delta S_k = 0, \qquad (6.18)$$

and the stationary condition is obtained as:

$$\frac{\partial \Pi}{\partial \boldsymbol{x}} = \sum_j 2w_j L_j \nabla L_j + \sum_k 2w_k S_k \nabla S_k = \mathbf{0}. \qquad (6.19)$$

Eq. (6.17)-(6.19) are just simple compositions of corresponding expressions related to the original FDM and the SSDM, which imply the potential ability of such expressions for extension.

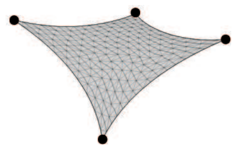


Figure 6.5: Membrane with Cables

Next, let us review form-finding methods which have been proposed to determin the forms of tensegrities. Particularly, let us examine the following two problems:

$$\begin{aligned}\Pi(\boldsymbol{x}, \lambda) = &\sum_j \frac{1}{2} k_j \left(L_j(\boldsymbol{x}) - \bar{L}_j\right)^2 \\ &+ \sum_k \lambda_k (L_k(\boldsymbol{x}) - \bar{L}_k) \rightarrow \text{stationary}, \qquad (6.20)\end{aligned}$$

$$\begin{aligned}\Pi(\boldsymbol{x}, \lambda) = &\sum_j w_j L_j^4(\boldsymbol{x}) \\ &+ \sum_k \lambda_k (L_k(\boldsymbol{x}) - \bar{L}_k) \rightarrow \text{stationary}, \qquad (6.21)\end{aligned}$$

where the first sum is taken for all the cables and the second is for all the struts.

In Ref. [8], Eq. (6.20) is proposed for the form-finding of tensegrities. In Eq. (6.20), k_j and $\bar{L}_j$ represent virtual stiffness and virtual initial length of the j-th cable respectively, which do not represent real material but define special (soft) material for form-finding analysis. Therefore, as discussed below, an appropriate set of $\bar{L}_j$ is needed. On the other hand, $\bar{L}_k$ represents just the objective length of the k-th strut. Fig. 6.6(a) shows an example of tensegrities which was obtained by solving Eq. (6.20) by the authors.

On the other hand, Eq. (6.21) is one of the stationary problems which was just proposed in this work. Fig. 6.6(b) shows an example of tensegrities given by solving Eq. (6.21).

With respect to the the second sums for the struts, there look no difference.

On the other hand, with respect to the first sums, which are for the cables, some differences can be recognized. They are, the powers and the terms that are powered. In addition, while the first sum of Eq. (6.20) looks an analogy of elastic energy of Hook's spring, the first sum of Eq. (6.21) looks different.

Then, let us see the **Principle of Virtual Works** corresponding to Eq. (6.20) and Eq. (6.21), i.e.

$$\delta w = \sum_j k_j \left(L_j - \bar{L}_j\right) \delta L_j + \sum_k \lambda_k \delta L_k = 0, \qquad (6.22)$$

$$\delta w = \sum 4 w_j L_j^3 \delta L_j + \sum_k \lambda_k \delta L_k = 0. \qquad (6.23)$$

Thus, it can be noticed that, in Eq. (6.22), the general forces $k_j\left(L_j - \overline{L}_j\right)$ which tend to produce small change of L_j are proportional to $(L_j - \overline{L}_j)$. Due to the fact that $\left(L_j - \overline{L}_j\right)$ can take negative numbers, some of the cables may become compression. Then, it can be noticed that an appropriate set of $\overline{L}_j$ is needed to ensure every $(L_j - \overline{L}_j)$ be positive.

For this purpose, one of the simplest ideas to determine each $\overline{L}_j$ in Eq. (6.20) for the cables is to set every $\overline{L}_j$ as 0. However, when every $\overline{L}_j$ are set to 0 in Eq. (6.20) or Eq. (6.22), some difficulties arise as mentioned in section 4. Then, to eliminate the difficulties, one of the simplest ideas is to alter the power of the term $\left(L_j - \overline{L}_j\right)$ to other numbers such as 4. Thus, the equations used in the extended FDM, such as Eq. (6.21) and Eq. (6.23), emerge.

In addition, the **Principle of Virtual Work** corresponding to Eq. (6.21) is also represented in the following form:

$$\delta w = \sum_j w_j \delta\left(L_j^4\right) + \sum_k \lambda_k \delta L_k = 0, \qquad (6.24)$$

which states that the extended force densities, i.e. $w_j = n_j/4L_j^3$, can be considered as general forces which act within the cables and tend to produce small change of L_j^4.

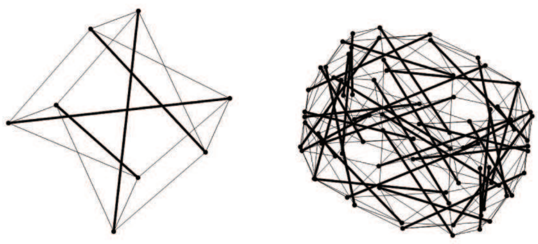
As a result of above discussion, a common feature which is shared by many form-finding methods have been found. By seeing the following above mentioned **Principle of Virtual Works**,

$$\delta w = \sum_j 2w_j L_j \delta L_j = 0, \qquad (6.25)$$

$$\delta w = \sum_j 2w_j S_j \delta S_j = 0, \qquad (6.26)$$

$$\delta w = \sum_j \delta S_j = 0, \qquad (6.27)$$

it can be noticed that the general forces which act within the elements or the members remain always positive.



(a) Result by Eq. (6.20) (b) Result by Eq. (6.21)

Figure 6.6: Form-Finding of Tensegrities

Finally, the stationary problems of functionals, the principle of virtual works and the stationary conditions using ∇ symbol, which were just compared in this section, are shown in Tab. 2 to 4 as an overview. By using those three expressions that are usually found in various problems of statics, the common features and the differences over various form-finding methods can be examined, as discussed in this section. Moreover, they also enable us to combine or extend the methods in natural ways.

7. Conclusions

In the first part of this work, the extended force density method was proposed. It enables us to carry out form-finding of prestressed structures that consist of combinations of both tension and compression members.

The existence of a variational principle in the FDM was pointed out and a functional that simply represents the FDM was proposed. Then, the FDM was extensively redefined by generalizing the formulation of the functional. Additionally, it was indicated that various functionals can be selected for form-finding of tension structures. Then, some form finding analyses of different types of tension structures were illustrated to show the potential ability of the extended FDM.

In the second part, various form-finding methods were reviewed and compared in the relation with the extended FDM. By using three types of expressions such as the principle of virtual work, which can be commonly found in general problems of statics and are also used in the description of the extended FDM, the common features and differences over different form-finding methods were examined.

Acknowledgments

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Table 2: Table of Stationary Problems

Force Density Method[2]		**Minimal Surface** e.g. [9]	
cables		membrane	
(a)	$\Pi = \sum wL^2 \rightarrow$ stationary.	(b)	$\Pi = \sum S \rightarrow$ stationary.
Surface stress density method[7]			
membrane		membrane with cables	
(c)	$\Pi = \sum wS^2 \rightarrow$ stationary.	(d)	$\Pi = \sum wL^2 + \sum wS^2 \rightarrow$ stationary.
Variational Method for Tensegrities[8]		**Extended Force Density Method** (Proposed)	
cables and struts		cables and struts	
(e)	$\Pi = \sum \frac{1}{2}k(L-\bar{L})^2 + \sum \lambda(L-\bar{L}) \rightarrow$ stationary.	(f)	$\Pi = \sum wL^4 + \sum \lambda(L-\bar{L}) \rightarrow$ stationary.

Table 3: Table of Principle of Virtual Works

Force Density Method[2]		**Minimal Surface** e.g. [9]	
cables		membrane	
(a)	$\delta w = \sum 2wL\delta L = 0$	(b)	$\delta w = \sum \delta S = 0$
Surface stress density method[7]			
membrane		membrane with cables	
(c)	$\delta w = \sum 2wS\delta S = 0$	(d)	$\delta w = \sum 2wL\delta L + \sum 2wS\delta S = 0$
Variational Method for Tensegrities[8]		**Extended Force Density Method** (Proposed)	
cables and struts		cables and struts	
(e)	$\delta w = \sum kL\delta L + \sum \lambda\delta L = 0$	(f)	$\delta w = \sum 4wL^3\delta L + \sum \lambda\delta L = 0$

Table 4: Table of Stationary Conditions

Force Density Method[2]		**Minimal Surface** e.g. [9]	
cables		membrane	
(a)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum 2wL\nabla L = \mathbf{0}$	(b)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum \nabla S = \mathbf{0}$
Surface stress density method[7]			
membrane		membrane with cables	
(c)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum 2wS\nabla S = \mathbf{0}$	(d)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum 2wL\nabla L + \sum 2wS\nabla S = \mathbf{0}$
Variational Method for Tensegrities[8]		**Extended Force Density Method** (Proposed)	
cables and struts		cables and struts	
(e)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum kL\nabla L + \sum \lambda\nabla L = \mathbf{0}$	(f)	$\frac{\partial\Pi}{\partial \boldsymbol{x}} = \sum 4wL^3\nabla L + \sum \lambda\nabla L = \mathbf{0}$

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Appendix A. Gradients

Appendix A.1. Gradient of Linear Element Length

Let p and q denote two nodes. Let

$$\boldsymbol{p} \equiv \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \boldsymbol{q} \equiv \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \quad \text{(A.1)}$$

represent the Cartesian coordinates of p and q.

The length of the line determined by p and q is given by

$$L\left(p_x,\ p_y,\ p_z,\ q_x,\ q_y,\ q_z\right) \quad \text{(A.2)}$$

$$\equiv \sqrt{(p_x-q_x)^2+(p_y-q_y)^2+(p_z-q_z)^2}. \quad \text{(A.3)}$$

If the gradient of L is defined by

$$\nabla L \equiv \left[\frac{\partial L}{\partial p_x}, \frac{\partial L}{\partial p_y}, \frac{\partial L}{\partial p_z}, \frac{\partial L}{\partial q_x}, \frac{\partial L}{\partial q_y}, \frac{\partial L}{\partial q_z}\right], \quad \text{(A.4)}$$

its components are as follows:

$$\nabla L = \left[\frac{p_x-q_x}{L}, \frac{p_y-q_y}{L}, \frac{p_z-q_z}{L}, \frac{q_x-p_x}{L}, \frac{q_y-p_y}{L}, \frac{q_z-p_z}{L}\right]. \quad \text{(A.5)}$$

Let us investigate δL, i.e.

$$\delta L \equiv \nabla L \cdot \begin{bmatrix} \delta\boldsymbol{p} \\ \delta\boldsymbol{q} \end{bmatrix}. \quad \text{(A.6)}$$

As shown in Fig. A.1, $\delta\boldsymbol{p}$ and $\delta\boldsymbol{q}$ are firstly projected to the line determined by p and q, then, δL is measured on the line.

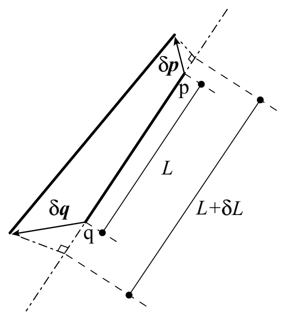

Figure A.1: Variation of Element Length

Appendix A.2. Gradient of Triangular Element Area

Let p, q, and r be three vertices. Let

$$\boldsymbol{p} \equiv \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \boldsymbol{q} \equiv \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \boldsymbol{r} \equiv \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}, \tag{A.7}$$

denote the Cartesian coordinates of p, q, and r.

The area of the triangle determined by p, q, and r is given by

$$S\left(p_x, \cdots, r_z\right) \equiv \frac{1}{2}\sqrt{\boldsymbol{N}\cdot\boldsymbol{N}}, \tag{A.8}$$

$$\left(\boldsymbol{N} \equiv (\boldsymbol{q}-\boldsymbol{p})\times(\boldsymbol{r}-\boldsymbol{p})\right). \tag{A.9}$$

If the gradient of S is defined by

$$\nabla S \equiv \left[\ \frac{\partial S}{\partial p_x},\ \ \frac{\partial S}{\partial p_y},\ \ \frac{\partial S}{\partial p_z},\ \ \cdots\ \ ,\frac{\partial S}{\partial r_z}\ \right], \tag{A.10}$$

its components are as follows:

$$\begin{aligned} \nabla S = \frac{1}{2}\boldsymbol{n}\cdot\Bigg[&(\boldsymbol{r}-\boldsymbol{q})\times\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \\ &,(\boldsymbol{p}-\boldsymbol{r})\times\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \\ &,(\boldsymbol{q}-\boldsymbol{p})\times\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\Bigg], \end{aligned} \tag{A.11}$$

where $\boldsymbol{n}$ is defined by

$$\boldsymbol{n} \equiv \frac{\boldsymbol{N}}{|\boldsymbol{N}|}. \tag{A.12}$$

Let us investigate δS, i.e.

$$\delta S = \frac{1}{2}\boldsymbol{n}\cdot\left((\boldsymbol{r}-\boldsymbol{q})\times\delta\boldsymbol{p} + (\boldsymbol{p}-\boldsymbol{r})\times\delta\boldsymbol{q} + (\boldsymbol{q}-\boldsymbol{p})\times\delta\boldsymbol{r}\right). \tag{A.13}$$

With respect to $\delta\boldsymbol{p}$, for example, when $\delta\boldsymbol{p}$ is orthogonal to the element, $(\boldsymbol{r}-\boldsymbol{q})\times\delta\boldsymbol{p}$ becomes orthogonal to $\boldsymbol{n}$, then δS vanishes (see Fig. A.2). On the other hand, when $\delta\boldsymbol{p}$ is parallel to the opposite side, $(\boldsymbol{r}-\boldsymbol{q})\times\delta\boldsymbol{p}$ vanishes, then δS vanishes. Therefore, only the component of $\delta\boldsymbol{p}$ which is parallel to the perpendicular line from p to the opposite side can produce δS. In other words, δS is measured on the plane determined by p, q, and r.

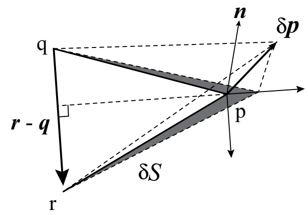

Figure A.2: Variation of Element Area

Appendix B. Some Remarks of Surface Area

Appendix B.1. Minimal Surfaces and Uniform Stress Surfaces

The surface area of a surface is given by

$$a = \int_a \mathrm{da}. \tag{B.1}$$

Here, da is called area element and defined by

$$\mathrm{da} \equiv \sqrt{\det g_{ij}}\, d\theta^1 d\theta^2, \tag{B.2}$$

where g_{ij} and $\left\{\theta^i, \theta^j\right\}$ are the Riemannian Metric and the local coordinate on the surface respectively.

Using Eq. (B.2), the variation of the surface area, δa, can be calculated and the result is as follows:

$$\delta a = \frac{1}{2}\int_a g^{ij}\delta g_{ij}\sqrt{\det g_{ij}}\, d\theta^1 d\theta^2, \tag{B.3}$$

$$\therefore \delta a = \frac{1}{2}\int_a g^{ij}\delta g_{ij}\mathrm{da}, \tag{B.4}$$

where g^{ij} is the inverse matrix of g_{ij}.

By the way, on a membrane, the 2nd *Piola-Kirchhoff* stress tensor and the *Green-Lagrange* strain tensor are defined by

$$\boldsymbol{S} \equiv \frac{\sqrt{\det g_{ij}}}{\sqrt{\det \bar{g}_{ij}}} T^i_{\cdot k} g^{kj}\, \bar{\boldsymbol{g}}_i \otimes \bar{\boldsymbol{g}}_j\,,\ \boldsymbol{E} \equiv \frac{1}{2}(g_{ij} - \bar{g}_{ij})\bar{\boldsymbol{g}}^i \otimes \bar{\boldsymbol{g}}^j, \tag{B.5}$$

where $T^i_{\cdot k}$ are the components of the Cauchy stress tensor. In addition, $\bar{\boldsymbol{g}}_i$, $\bar{\boldsymbol{g}}^i$, $\bar{g}_{ij}$ are the dual bases and the Riemannian metric defined on a reference configuration.

Then, the **Principle of Virtual Work** for membranes is expressed as:

$$\delta w = \int_{\bar{a}} t\boldsymbol{S} : \delta\boldsymbol{E}\,\mathrm{d}\bar{\mathrm{a}} \tag{B.6}$$

where $\mathrm{d}\bar{\mathrm{a}}$, $\bar{a}$ are related to the reference configuration, and t denotes the thickness. Eq. (B.6) reduces to the following form:

$$\delta w = \int_a t T^i_{\cdot k} g^{kj}\delta g_{ij}\mathrm{da}, \tag{B.7}$$

which does not depend on the reference configuration.

Because Eq. (B.4) can be transformed into the following form:

$$\delta a = \int_a \delta^i_{\cdot k} g^{kj}\delta g_{ij}\mathrm{da}, \tag{B.8}$$

when t and $T^{i}_{\cdot k}$ are uniform on the surface and when $T^{i}_{\cdot k} = \hat{\sigma}\delta^{i}_{\cdot k}$, where $\hat{\sigma}$ is also uniform, then

$$\delta w = t\hat{\sigma}\delta a \quad \therefore \delta w = 0 \Leftrightarrow \delta a = 0, \tag{B.9}$$

which is a simple demonstration of the essential identity of uniform stress surfaces and minimal surfaces.

Appendix B.2. Galerkin Method for Minimal Surface

When the form of a surface is represented by n-independent parameters such as $\boldsymbol{x} = \left[\begin{array}{ccc} x_1 & \cdots & x_n \end{array} \right]^T$, an approximation of

$$\delta a = \int_a g^{ij} \delta g_{ij} \mathrm{da} = 0 \tag{B.10}$$

can be obtained by the *Galerkin method* and it is as follows:

$$\delta \tilde{a} = \left(\int_a g^{ij} \nabla g_{ij} \mathrm{da} \right) \cdot \delta \boldsymbol{x} = 0, \tag{B.11}$$

where ∇ is the gradient operator defined by

$$\nabla f \equiv \left[\begin{array}{ccc} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{array} \right] \tag{B.12}$$

and $\delta \boldsymbol{x}$ is the variation of $\boldsymbol{x}$, or, just an arbitrary column vector.

When the form is discretized by m elements, the integral can be divided into m independent integrals. Hence

$$\delta \tilde{a} = \left(\sum_j \int_a g^{\alpha\beta} \nabla g_{\alpha\beta} \mathrm{da} \bigg|_j \right) \cdot \delta \boldsymbol{x}, \tag{B.13}$$

where j is the index of each element.

In each element, remembering the relation of

$$\int_a g^{\alpha\beta} \delta g_{\alpha\beta} \mathrm{da} \bigg|_j = \delta \int_a \mathrm{da} \bigg|_j, \tag{B.14}$$

the following transformation is also correct:

$$\int_a g^{\alpha\beta} \nabla g_{\alpha\beta} \mathrm{da} \bigg|_j = \nabla \int_a \mathrm{da} \bigg|_j, \tag{B.15}$$

due to the fact that δ symbol is originally defined by $\frac{\partial}{\partial \epsilon}$ when ϵ is the assigned one-parameter to represent the change of the form.

Therefore, when S_j is defined as a function to give j-th element area, i.e.

$$S_j \equiv \int_a \mathrm{da} \bigg|_j, \tag{B.16}$$

then

$$\delta \tilde{a} = 0 \Leftrightarrow \left(\sum_j \nabla S_j \right) \cdot \delta \boldsymbol{x} = 0 \Leftrightarrow \sum_j \nabla S_j = \mathbf{0}, \tag{B.17}$$

which is the stationary condition of

$$\Pi(\boldsymbol{x}) = \sum_j S_j(\boldsymbol{x}). \tag{B.18}$$